Problem 1. (10pts)

a) Find the differential cross section for scattering, in the Born approximation, of a particle of mass $m$ from a repulsive square potential $V(r) = V_0$ for $r < r_0$ and $V(r) = 0$ for $r > r_0$, with $V_0$ positive.

For the constant potential described above we can evaluate the scattering amplitude in the Born approximation using the expression:

$$f_k^{(B)}(\theta) = -\frac{2\mu}{\hbar^2 q} \int_0^\infty r' V(r') \sin(qr')dr'$$  \hspace{1cm} (1)

where $V(r') = V_0$ for $r < r_0$. There is no contribution to the amplitude for $r > r_0$ therefore

$$f_k^{(B)}(\theta) = -\frac{2\mu V_0}{\hbar^2 q} \int_0^{r_0} r' \sin(qr')dr'$$  \hspace{1cm} (2)

We will use integration by parts, using

$$\int u dv = uv - \int v du,$$  \hspace{1cm} (3)

to evaluate the above integral. We will set $u = r'$ and $dv = \sin(qr')dr'$ thus $du = dr'$ and $v = -1/q \cos(qr')$

$$\int_0^{r_0} r' \sin(qr')dr' = -\frac{r'}{q} \cos(qr')|_0^{r_0} + \int_0^{r_0} \frac{1}{q} \cos(qr')dr'$$  \hspace{1cm} (4)

$$= -\frac{r_0}{q} \cos(qr_0) + \frac{1}{q^2} \sin(qr_0)$$  \hspace{1cm} (5)

Leading to the scattering amplitude

$$f_k^{(B)}(\theta) = \frac{2\mu V_0}{\hbar^2 q^2} \left( r_0 \cos(qr_0) - \frac{\sin(qr_0)}{q} \right)$$  \hspace{1cm} (6)

and to the differential cross section

$$\frac{d\sigma}{d\Omega}(\theta) = \left( \frac{2\mu V_0 r_0^2}{\hbar^2} \right)^2 \frac{\sin(qr_0) - qr_0 \cos(qr_0)^2}{(qr_0)^6}$$  \hspace{1cm} (7)
b) Calculate the total cross section for the low-energy s-wave scattering of a particle of mass $m$ from an attractive square potential $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, with $V_0$ positive.

Here, at low energy, the cross section is dominated by the s partial wave and therefore $l = 0$ partial wave. The radial part of the Schrödinger equation is given by

$$\frac{-\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} - V_0 u(r) = E u(r) \quad \text{for} \quad r > r_0$$ (8)

$$\frac{-\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} = E u(r) \quad \text{for} \quad r < r_0$$ (9)

where $u(r) = r R(r)$. The solutions of these equations for positive energy states are:

$$u(r) = \begin{cases} u_1(r) = A \sin k_1 r & \text{for} \quad r < r_0 \\ u_1(r) = B \sin (k_2 r + \delta_0) & \text{for} \quad r > r_0 \end{cases}$$ (11)

where

$$k_1 = \sqrt{\frac{2m(E + V_0)}{\hbar^2}} \quad \text{and} \quad k_2 = \sqrt{\frac{2mE}{\hbar^2}}$$ (12)

The continuity of $u(r)$ and its first derivative $du(r)/dr$ at $r = r_0$ leads to the following equations

$$u_1(r_0) = u_2(r_0) \Rightarrow A \sin k_1 r_0 = B \sin (k_2 r_0 + \delta_0)$$ (13)

$$u_1'(r_0) = u_2'(r_0) \Rightarrow Ak_1 \cos k_1 r_0 = Bk_2 r_0 \cos (k_2 r_0 + \delta_0)$$ (14)

taking the ratio of equation (13) over (14) we find

$$\frac{1}{k_1} \tan k_1 r_0 = \frac{1}{k_2} \tan (k_2 r_0 + \delta_0)$$ (15)

Since to determine the total cross section we are interested in the expression

$$\sigma_0 = \frac{4\pi}{k_1^2} \sin^2 \delta_0$$ (16)

we want to solve for $\sin^2 \delta_0$ using equation (15). First we express $\tan (k_2 r_0 + \delta_0)$ as

$$\tan (k_2 r_0 + \delta_0) = \frac{\tan k_2 r_0 + \tan \delta_0}{1 - \tan k_2 r_0 \tan \delta_0}$$ (17)

Thus

$$\tan \delta_0 = \frac{k_2 \tan k_1 r_0 - k_1 \tan k_2 r_0}{k_1 + k_2 \tan k_1 r_0 \tan k_2 r_0}$$ (18)

and since

$$\sin^2 \delta_0 = \frac{1}{1 + 1/\tan^2 \delta_0}$$ (19)

2
we can write
\[ \sin^2 \delta_0 = \left[ 1 + \left( \frac{k_1 + k_2 \tan k_1 r_0 \tan k_2 r_0}{k_2 \tan k_1 r_0 - k_1 \tan k_2 r_0} \right)^2 \right]^{-1} \]
and consequently the total cross section as
\[ \sigma_0 = \frac{4\pi}{k_1^2} \left[ 1 + \left( \frac{k_1 + k_2 \tan k_1 r_0 \tan k_2 r_0}{k_2 \tan k_1 r_0 - k_1 \tan k_2 r_0} \right)^2 \right]^{-1} \]

Problem 2. (5pts)

Quarks carry spin 1/2. Three quarks bind together to make a baryon (such as the proton or the neutron); two quarks (or more precisely a quark and an antiquark) bind together to make a meson (such as the pion or the kaon). Assume the quarks are in the ground state (so the orbital angular momentum is zero).

a) What spins are possible for baryons?

Given the rules of addition of angular momenta
we first add 2 quarks spins \( s_{1,2} = 1/2 \)
\[ |s_1 - s_2| \leq S_T \leq s_1 + s_2 \Rightarrow S_T = 0 \text{ or } S_T = 1 \] (22)
Thus the possibles values for the total spin of 2 quarks in this case is 0 or 1 assuming there is no orbital angular momentum. Now, we add the third quark spin \( s_3 = 1/2 \).
\[ |S_T - s_3| \leq S_B \leq S_T + s_3 \Rightarrow S_B = \frac{1}{2} \text{ or } S_B = \frac{3}{2} \] (23)
The possibles values are then \( S_B = 1/2 \) in one case or \( S_T = 1/2 \) or \( 3/2 \) in the other case. So in summary baryons can have spin 3/2 or spin 1/2 (and the latter can be achieved in two distinct ways). In fact light baryons do carry spin 1/2 (proton, neutron, etc) or 3/2 (\( \Delta, \Omega \), etc)

b) What spins are possible for mesons?

For mesons we only need to add 2 spins \( s_{1,2} = 1/2 \), one for the quark and the other for the antiquark.
\[ |s_1 - s_2| \leq S_M \leq s_1 + s_2 \Rightarrow S_M = 0 \text{ or } S_M = 1 \] (24)
Thus the possibles values for the total spin of a quark and antiquark in this case are 0 or 1 assuming there is no orbital angular momentum. In summary mesons can have spin 0 or spin 1. In fact light mesons do carry spin 0 (\( \pi \)'s, \( k \)'s etc) or 1 (\( \rho \)'s, \( \omega \)'s, etc)

Problem 3. (5pts)

The electron in a hydrogen atom occupies the combined spin and position state
\[ R_{21}(r) \left( \sqrt{\frac{1}{3}} Y_0^0(\theta, \phi) |+ > + \sqrt{\frac{2}{3}} Y_1^1(\theta, \phi) |− > \right) \] (25)
or written fully in the Hilbert space notation of the basis tensor product of \{nlm\} \otimes \{\pm\}

\[
\sqrt{\frac{1}{3}}|2\uparrow 0\rangle \otimes |+\rangle + \sqrt{\frac{2}{3}}|2\uparrow 1\rangle \otimes |\rangle
\]

(26)

Let \( \vec{J} = \vec{L} + \vec{S} \) be the total angular momentum

\[ a) \text{ If you measured } \hat{J}_2 \text{ and } \hat{J}_z \text{ what values might you get and with what probability for each.} \]

Clearly the total angular momentum is conserved in the hydrogen atom and given the state occupied by the electron we can already state that the value of \( L = 1 \) and the value of \( S = 1/2 \). Therefore the possible values of \( J^2 \) that can be measured are

\[ |L - S| \leq J \leq L + S \Rightarrow J = 1/2 \text{ or } J = 3/2 \]

(27)

Therefore if we measure \( \hat{J}_2 \) the results are the eigenvalues \( 3/4 \hbar^2 \) and \( 15/4 \hbar^2 \) The eigenvalues of \( J_z \) are comprised between \(-J\) and \( +J \), thus in this case for

\[ J = 1/2 \text{ the possible results for } \hat{J}_z \text{ are } -\hbar/2 \text{ and } +\hbar/2 \]

(28)

\[ J = 3/2 \text{ the possible results for } \hat{J}_z \text{ are } -3/2\hbar, -\hbar/2, +\hbar/2 \text{ and } +3/2\hbar \]

(29)

To evaluate the probabilities we need to project each \(|J,M\rangle\) onto the state by equation (26).

Using the Clebsch-Gordon coefficients provided table we can write the states in the basis of \{\( \hat{L}^2, \hat{S}^2, \hat{J}^2, \hat{J}_z \)\} in terms of that of \{\( \hat{L}^2, \hat{L}_z, \hat{S}^2, \hat{S}_z \)\} basis.

\[ |3/2, +3/2\rangle = |2\uparrow 1 + 1\rangle \otimes |+\rangle \]

(31)

\[ |3/2, +1/2\rangle = \sqrt{\frac{1}{3}}|2\uparrow 1 + 1\rangle \otimes |\rangle + \sqrt{\frac{2}{3}}|2\uparrow 0\rangle \otimes |+\rangle \]

(32)

\[ |3/2, -1/2\rangle = \sqrt{\frac{2}{3}}|2\uparrow 1 - 1\rangle \otimes |+\rangle + \sqrt{\frac{1}{3}}|2\uparrow 0\rangle \otimes |\rangle \]

(33)

\[ |3/2, -3/2\rangle = |2\uparrow 1 - 1\rangle \otimes |\rangle \]

(34)

\[ |1/2, +1/2\rangle = \sqrt{\frac{2}{3}}|2\uparrow 1 + 1\rangle \otimes |\rangle - \sqrt{\frac{1}{3}}|2\uparrow 0\rangle \otimes |+\rangle \]

(35)

\[ |1/2, -1/2\rangle = -\sqrt{\frac{2}{3}}|2\uparrow 1 - 1\rangle \otimes |+\rangle + \sqrt{\frac{1}{3}}|2\uparrow 0\rangle \otimes |\rangle \]

(36)

The probability to find \(+15/4\hbar^2\) when measuring \( \hat{J}_2 \) is given by

\[
P(+15/4\hbar^2) = \left| \langle 3/2, M| \left( \sqrt{\frac{1}{3}}|2\uparrow 0\rangle \otimes |+\rangle \right) + \sqrt{\frac{2}{3}}|2\uparrow 1\rangle \otimes |\rangle \right|^2
\]

(40)

The non vanishing term is that when \( M = 1/2 \) namely
\[
\begin{align*}
&= \left| \left( \sqrt{\frac{1}{3}} |2 1 +1 \rangle \otimes \langle -| + \sqrt{\frac{2}{3}} |2 1 0 \rangle \otimes \langle +| \right) \left( \sqrt{\frac{1}{3}} |2 1 0 \rangle \otimes \langle +| + \sqrt{\frac{2}{3}} |2 1 1 \rangle \otimes \langle -| \right) \right|^2 \\
&= \left| \sqrt{\frac{1}{3}} \sqrt{\frac{2}{3}} + \sqrt{\frac{2}{3}} \sqrt{\frac{1}{3}} \right|^2 = \left| \frac{2\sqrt{2}}{3} \right|^2 = \frac{8}{9}
\end{align*}
\] 

(41)

The probability to find \(+3/4\hbar^2\) when measuring \(\hat{J}^2\) is given by

\[
P(+3/4\hbar^2) = \langle 1/2, M | \left( \sqrt{\frac{1}{3}} |2 1 0 \rangle \otimes \langle +| + \sqrt{\frac{2}{3}} |2 1 1 \rangle \otimes \langle -| \right) \rangle^2
\]

(42)

The non vanishing term is that when \(M = 1/2\) namely

\[
\begin{align*}
&= \left| \left( \sqrt{\frac{1}{3}} |2 1 +1 \rangle \otimes \langle -| - \sqrt{\frac{1}{3}} |2 1 0 \rangle \otimes \langle +| \right) \left( \sqrt{\frac{1}{3}} |2 1 0 \rangle \otimes \langle +| + \sqrt{\frac{2}{3}} |2 1 1 \rangle \otimes \langle -| \right) \right|^2 \\
&= \left| \sqrt{\frac{2}{3}} \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} \sqrt{\frac{1}{3}} \right|^2 = \left| \frac{1}{3} \right|^2 = \frac{1}{9}
\end{align*}
\]

(43)

For the measurement of \(\hat{J}_z\) if the system is prepared in the state of equation (26) only one value is obtained namely \(+\hbar/2\) with probability of \(P=1\).

Useful trigonometric relations:

\[
\sin^2 \alpha = \frac{1}{1 + 1/\tan^2 \alpha}
\]

(44)

\[
\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\]

(45)

(46)
43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND D FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

**Notation:** \( j \) \( m \) \( J \) \( M \) \( m' \) \( m'' \)

\[
\begin{align*}
1 \times 1/2 & = 1/2 \times 1/2 \\
& = \begin{array}{cccc}
1 & 1/2 & 1/2 & 1/2 \\
+1/2 & 1/2 & 1/2 & 1/2 \\
-1/2 & 1/2 & 1/2 & 1/2 \\
-1/2 & -1/2 & 1/2 & 1/2 \\
+1/2 & -1/2 & 1/2 & 1/2 \\
+1/2 & 1/2 & 1/2 & 1/2 \\
-1/2 & -1/2 & 1/2 & 1/2 \\
-1/2 & 1/2 & 1/2 & 1/2 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
Y_1^0 & = \sqrt{3/4} \cos \theta \\
Y_1^1 & = \sqrt{3/4} \sin \theta e^{i\phi} \\
Y_2^0 & = \sqrt{3/4} \left[ \cos^2 \theta - 1/2 \right] \\
Y_2^1 & = \sqrt{3/4} \sin \theta \cos \theta e^{i\phi} \\
Y_2^2 & = \sqrt{3/4} \sin^2 \theta e^{i2\phi} \\
\end{align*}
\]

\[
\begin{align*}
2 \times 1/2 & = 1/2 \times 1/2 \\
& = \begin{array}{cccc}
1/2 & 1/2 & 1/2 & 1/2 \\
+1/2 & 1/2 & 1/2 & 1/2 \\
-1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & 1/2 & 1/2 \\
-1/2 & -1/2 & 1/2 & 1/2 \\
-1/2 & 1/2 & 1/2 & 1/2 \\
+1/2 & -1/2 & 1/2 & 1/2 \\
+1/2 & 1/2 & 1/2 & 1/2 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
3 \times 2/1 & = 3\times 2/1 \\
& = \begin{array}{cccc}
2/1 & 2/1 & 2/1 & 2/1 \\
+1/2 & 2/1 & 2/1 & 2/1 \\
1/2 & -1/2 & 2/1 & 2/1 \\
-1/2 & 1/2 & 2/1 & 2/1 \\
1/2 & 1/2 & 2/1 & 2/1 \\
-1/2 & -1/2 & 2/1 & 2/1 \\
-1/2 & 1/2 & 2/1 & 2/1 \\
+1/2 & -1/2 & 2/1 & 2/1 \\
+1/2 & 1/2 & 2/1 & 2/1 \\
\end{array}
\end{align*}
\]

**Figure 43.1:** The sign convention is that of Wigner (Group Theory, Academic Press, New York, 1959), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge Univ. Press, New York, 1953), Ross (Elementary Theory of Angular Momentum, Wiley, New York, 1957), and Cohen (Tables of the Clebsch-Gordan Coefficients, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

Figure 1: Table of Clebsh-Gordon coefficients extracted from the data particle book [http://www-pdg.lbl.gov](http://www-pdg.lbl.gov)